

## Lecture 11 (March 7, 2016)

Linearization  $\dot{x} = Ax$

$A$  is a "Hurwitz" matrix (stability matrix) if  $\operatorname{Re} \lambda_i < 0$ .

The origin of  $\dot{x} = Ax$  is a.s. iff  $A$  is Hurwitz.

Let  $V = x^T Px$ , with  $P > 0$ . Then,

$$\dot{V} = x^T P \dot{x} + \dot{x}^T P x = x^T (PA + A^T P)x = -x^T Qx$$

where  $Q$  is symmetric & satisfies the "Lyapunov equation" (L.E.) :

$$PA + A^T P = -Q \quad (\text{L.E.})$$

**Theorem 4.6.**  $A$  is Hurwitz iff for any  $Q > 0, \exists P > 0$  that satisfies Lyapunov equation. If  $A$  is Hurwitz,  $P$  is unique solution of (L.E.).

Positive definiteness requirement on  $Q$  can relax to a positive semi-definite matrix of the form  $Q = c^T c$  where  $(A, c)$  is observable, i.e.  $C e^{At} x = 0 \quad \forall t \text{ iff } x = 0$ . (Exe 4.21)

Proof sufficiency from Thm 4.1. ( $\dot{V} = -x^T Qx < 0$ )

Assume  $A$  is Hurwitz ( $\operatorname{Re} \lambda_i < 0$ , for  $A$ ).

choice of  $P$ : Let  $P = \int_0^\infty e^{At} Q e^{A^T t} dt \quad (*) \quad (\text{sum of } t^{k-1} e^{\lambda_i t})$

- $P$  is symmetric. ( $Q > 0 \Rightarrow Q = BB^T$ )

- $P$  is positive definite. suppose  $P$  is not positive definite. Then

$$\exists x \neq 0 \text{ s.t. } x^T Px = 0 \Rightarrow \int_0^\infty \underbrace{x^T e^{At}}_{y^T} \underbrace{Q e^{A^T t}}_{> 0} \underbrace{x}_{y} dt = 0$$

$$\Rightarrow e^{At} x = 0 \quad \forall t \geq 0$$

$$\Rightarrow x = 0 \quad \text{contradiction.}$$

Plug  $P$  in L.E.

$$\begin{aligned}
 PA + A^T P &= \int_0^\infty e^{At} Q e^{At} A dt + \int_0^\infty A^T e^{At} Q e^{At} dt \\
 &= \int_0^\infty \frac{d}{dt} e^{At} Q e^{At} dt = e^{At} Q e^{At} \Big|_0^\infty \stackrel{(Re \lambda_i < 0)}{=} -Q
 \end{aligned}$$

$\Rightarrow P$  is a solution of L.E. Next we show that  $P$  is unique.

Suppose  $\tilde{P} \neq P$  is another solution of L.E.

$$\Rightarrow (P - \tilde{P}) A + A^T (P - \tilde{P}) = 0$$

Pre & Post multiply by  $e^{At}$  &  $e^{At}$  to get

$$0 = e^{At} [(P - \tilde{P}) A + A^T (P - \tilde{P})] e^{At} = \frac{d}{dt} \{ e^{At} (P - \tilde{P}) e^{At} \}$$

$$\Rightarrow e^{At} (P - \tilde{P}) e^{At} = \text{constant } \forall t.$$

Since at  $t=0$ ,  $e^{A0} = I$ , we have

$$P - \tilde{P} = e^{At} (P - \tilde{P}) e^{At} \xrightarrow[\text{as } t \rightarrow \infty]{0} 0 \Rightarrow P = \tilde{P}.$$

Usefulness of this Lyapunov function:

Idea is that for any Hurwitz matrix  $A$ , can construct a Lyapunov function. This then becomes useful for studying stability of linear systems with possibly non-linear perturbation.

Lyapunov's indirect method

Theorem 4.7 (related to Hartman-Grobman Thm)

Let  $x=0$  be an eq. pt. for  $\dot{x}=f(x)$ , where  $f: D \rightarrow \mathbb{R}^n$  is  $C^1$  &  $D$  is a neighborhood of the origin. Let  $A = \frac{\partial f}{\partial x}(x)|_{x=0}$ . Then

- 1) The origin is a.s. if  $A$  is Hurwitz.

2. The origin is unstable if  $\operatorname{Re} \lambda_i > 0$  for one or more of the eigenvalues of  $A$ .

Proof ① First check that

$$f(x) = Ax + g(x) \quad \& \quad \frac{\|g(x)\|}{\|x\|} \rightarrow 0 \text{ as } \|x\| \rightarrow 0.$$

Suppose  $A$  is Hurwitz. Let  $Q$  be positive definite, symmetric matrix and  $P$  the unique solution of (L.E.) Let  $V(x) = x^T P x$ . Then

$$\begin{aligned}\dot{V} &= x^T P \dot{x} + \dot{x}^T P x \\ &= x^T P f(x) + f^T(x) P x \\ &= x^T P (Ax + g(x)) + (x^T A^T + g^T(x)) P x \\ &= x^T (PA + A^T P)x + x^T Pg(x) + g^T(x) P x \\ &= -x^T Q x + x^T Pg(x) + g^T(x) P x\end{aligned}$$

Since  $\frac{\|g(x)\|}{\|x\|} \rightarrow 0$  as  $\|x\| \rightarrow 0$ , for any  $\delta > 0$ ,  $\exists r > 0$  s.t.

$$\|g(x)\| < \delta \|x\|, \quad \forall \|x\| < r$$

$$\begin{aligned}\therefore \dot{V} &< -x^T Q x + \|2x^T Pg(x)\| \\ &\leq -x^T Q x + 2\|x\|\|P\|\|g(x)\|, \quad \forall \|x\| < r \\ &< -x^T Q x + 2\delta\|P\|\|x\|^2 \\ &< -\lambda_{\min}(Q)\|x\|^2 + 2\delta\|P\|\|x\|^2 \\ &= -(\lambda_{\min}(Q) - 2\delta\|P\|)\|x\|^2\end{aligned}$$

Pick  $\delta$  s.t.  $\lambda_{\min}(Q) > 2\delta\|P\|$

Then  $\dot{V} < 0$ , for  $\|x\| < r$   $\Rightarrow 0$  a.s.  
 $x \neq 0$

② Find non-singular  $T$  s.t.  $T^{-1}AT = \begin{pmatrix} -A_1 & 0 \\ 0 & A_2 \end{pmatrix}$  ( $A_i$ : Hurwitz)

Change coordinates:  $z = Tx = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$

$$\Rightarrow \dot{z} = T\dot{x} = TAT^{-1}z + Tg(T^{-1}z)$$

$$\dot{z}_1 = -A_1 z_1 + \tilde{g}_1(z)$$

$$\dot{z}_2 = A_2 z_2 + \tilde{g}_2(z)$$

where for any  $\gamma > 0$ ,  $\|\tilde{g}_i(z)\| < \gamma \|z\|$ ,  $\forall \|z\| < r$ ,  $i=1,2$

To show  $z=0$  is unstable, (e.g.  $x=0$ ), Apply Thm 4.3.

Let  $Q_1 > 0$ ,  $Q_2 > 0$  and  $P_1 > 0$ ,  $P_2 > 0$  where

$$P_i A_i + A_i^T P_i = -Q_i \quad i=1,2$$

$$\text{Let } V(z) = z_1^T P_1 z_1 - z_2^T P_2 z_2 = z^T \begin{bmatrix} P_1 & 0 \\ 0 & -P_2 \end{bmatrix} z$$

Clearly when  $z=0$ ,  $V(z) > 0$  at points arbitrary close to the origin.

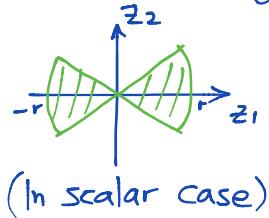
Let  $\mathcal{U} = \{z \in \mathbb{R}^n \mid \|z\| \leq r \text{ and } V(z) > 0\}$ .

( $z_1$  &  $z_2$  are vectors  $\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \in \mathbb{R}^n$ )

In  $\mathcal{U}$  we want to show that  $\dot{V} > 0$ :

$$\begin{aligned} \dot{V} &= z_1^T P_1 \dot{z}_1 + \dot{z}_1^T P_1 z_1 - (z_2^T P_2 \dot{z}_2 + \dot{z}_2^T P_2 z_2) \\ &= z_1^T P_1 (-A_1 z_1 + \tilde{g}_1(z)) + (-z_1^T A_1^T + \tilde{g}_1^T(z)) P_1 z_1 \\ &\quad - \left\{ z_2^T P_2 (A_2 z_2 + \tilde{g}_2(z)) + (z_2^T A_2^T + \tilde{g}_2^T(z)) P_2 z_2 \right\} \\ &= -z_1^T (P_1 A_1 + A_1^T P_1) z_1 - z_2^T (P_2 A_2 + A_2^T P_2) z_2 \\ &\quad + 2 z_1^T P_1 \tilde{g}_1(z) - 2 z_2^T P_2 \tilde{g}_2(z) \\ &= z_1^T Q_1 z_1 + z_2^T Q_2 z_2 + 2 z^T \begin{pmatrix} P_1 \tilde{g}_1(z) \\ -P_2 \tilde{g}_2(z) \end{pmatrix} \\ &\geq \lambda_{\min}(Q_1) \|z_1\|^2 + \lambda_{\min}(Q_2) \|z_2\|^2 - 2 \|z\| \left( \|P_1 \tilde{g}_1(z)\|^2 + \|P_2 \tilde{g}_2(z)\|^2 \right)^{1/2} \\ &\geq \min \{ \lambda_{\min}(Q_1), \lambda_{\min}(Q_2) \} \|z\|^2 - 2 \|z\| \left( \sum_{i=1,2} \|P_i\|^2 \|\tilde{g}_i\|^2 \right)^{1/2} \\ &\geq \alpha \|z\|^2 - 2 \|z\| \underbrace{\left( 2 \max \{ \|P_1\|, \|P_2\| \}^2 \beta^2 \|z\|^2 \right)^{1/2}}_{\beta^2} \\ &= (\alpha - 2\sqrt{2}\beta\gamma) \|z\| \end{aligned}$$

Choose  $\gamma$  s.t.  $\alpha - 2\sqrt{2}\beta\gamma > 0$ . Then  $\dot{V} > 0$  in  $\mathcal{U}$ . ■



## Stabilization via Linearization

$$\dot{x} = f(x, u), \quad f(0, 0) = 0$$

and  $f(x, u) : C^1$  in a domain  $D_x \times D_u \subset \mathbb{R}^n \times \mathbb{R}^p$  that contains the origin ( $x=0, u=0$ )

Problem: Design a state feedback control law  $u = \gamma(x)$  to stabilize the system.

Linearization about the origin gives  $\dot{x} = Ax + Bu$

$$A = \left. \frac{\partial f}{\partial x}(x, u) \right|_{x=0, u=0}; \quad B = \left. \frac{\partial f}{\partial u}(x, u) \right|_{x=0, u=0}$$

## Stabilizing feedback

Assume  $(A, B)$  is controllable (or at least stabilizable). Design  $K$  s.t.  $A - BK$  is Hurwitz. Let  $u = -Kx$ . Then, the closed loop system is  $\dot{x} = f(x, -Kx)$ . Linearization about  $x=0$  gives

$\dot{x} = (A - BK)x$ . By Thm 4.7. Origin is now a.s. (In fact, by Thm 4.13 is exponentially stable.)

Can even find Lyapunov function to prove it by choosing  $Q > 0$  & solving for  $P > 0$  from L.E.  $P(A - BK) + (A - BK)^T P = -Q$

Then  $V = x^T Px$ . Can even use this to estimate (R.A).